

Ancient Greek Problem: Can we trisect an angle with ruler and compass?

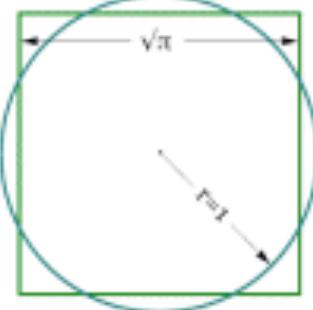

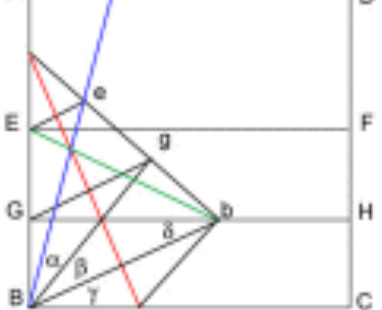
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MATH 3052 – Exploring Geometries

Abstract & Research goals





Ancient Greek Mathematic is considered to be the most important and basic foundation of modern Mathematics. Many great mathematicians have discovered theorems and formulas that is fundamental to modern mathematics. For example, everyone have learned Pythagorean theorem since elementary school. Euclid's Elements is known to be the greatest textbook in history of Math until 12th Century. However, there were many other problems couldn't be solved. One of them is angle trisection. Trisection is the process of dividing something into three equal parts. Our goal was to prove the impossibility of trisection of an angle using **ruler and compass only**.

Historical Background

There were three ancient Greek Problems using ruler and compass.

 squaring the circle	 doubling the cube	 trisecting the angle
<p>construct a square with an area exactly equal to that of a given circle</p> <p>proved impossible by Ferdinand von Lindeman in 1882</p>	<p>construct (duplicate) a cube with exactly twice the volume of a given cube</p> <p>proved impossible by Pierre Wantzel in 1837</p>	<p>construct an angle exactly one-third of any given angle</p> <p>proved impossible by Pierre Wantzel in 1837</p>

You may ask “Why solving using ruler and compass only?” The reason is that Euclid's The Element was taught as a primary textbook of geometry and number theory. The first three postulates (statements that is accepted as true without proof) attributed by Plato, has become the main integral part of Greek Geometry. Thus, constructing geometric figures using only a straightedge (ruler) and compass (referred as plane methods) had become main problems. Thus, many amateur and intellectual mathematicians tried to solve it.

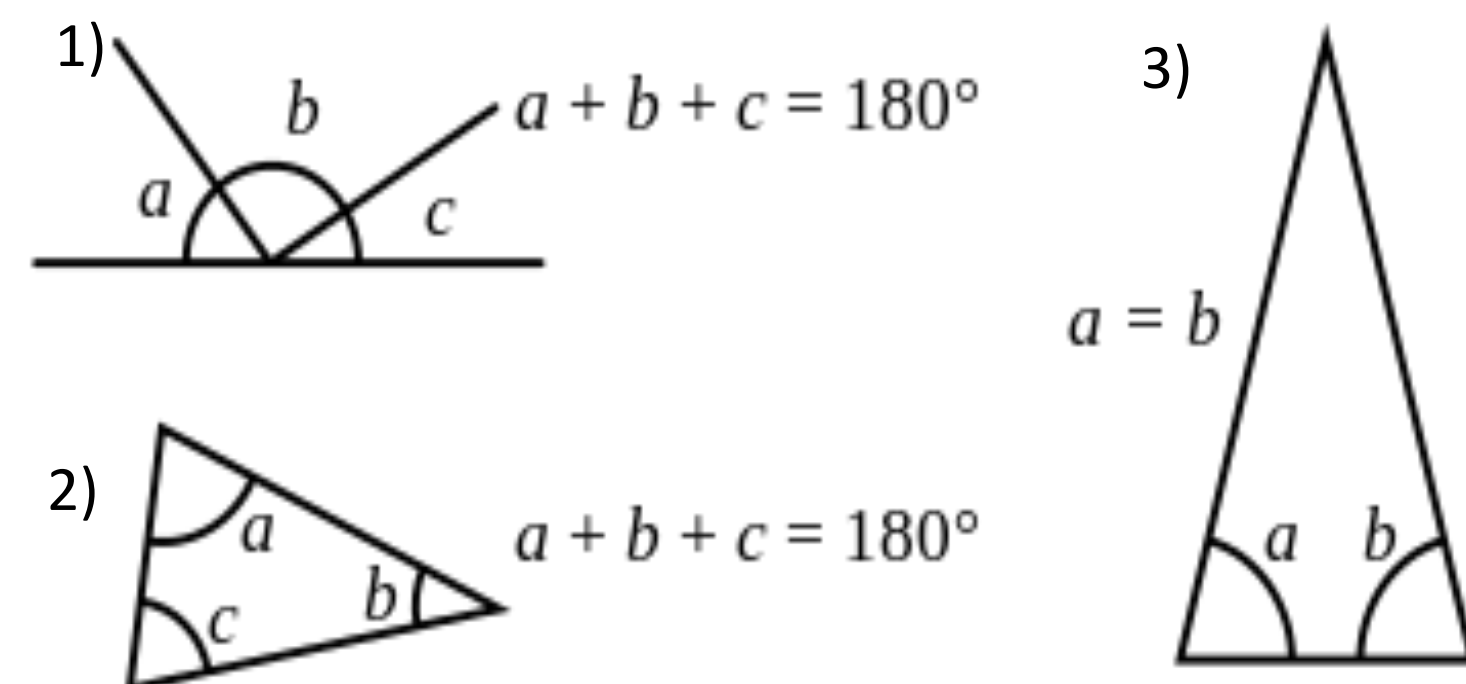
 I.	 II.	 III.	 IV.
<p>I. A straight line can be drawn between any two points II. A finite straight line can be extended infinitely III. A circle can be drawn with any center point and a line segment</p>			

Euclid's postulates

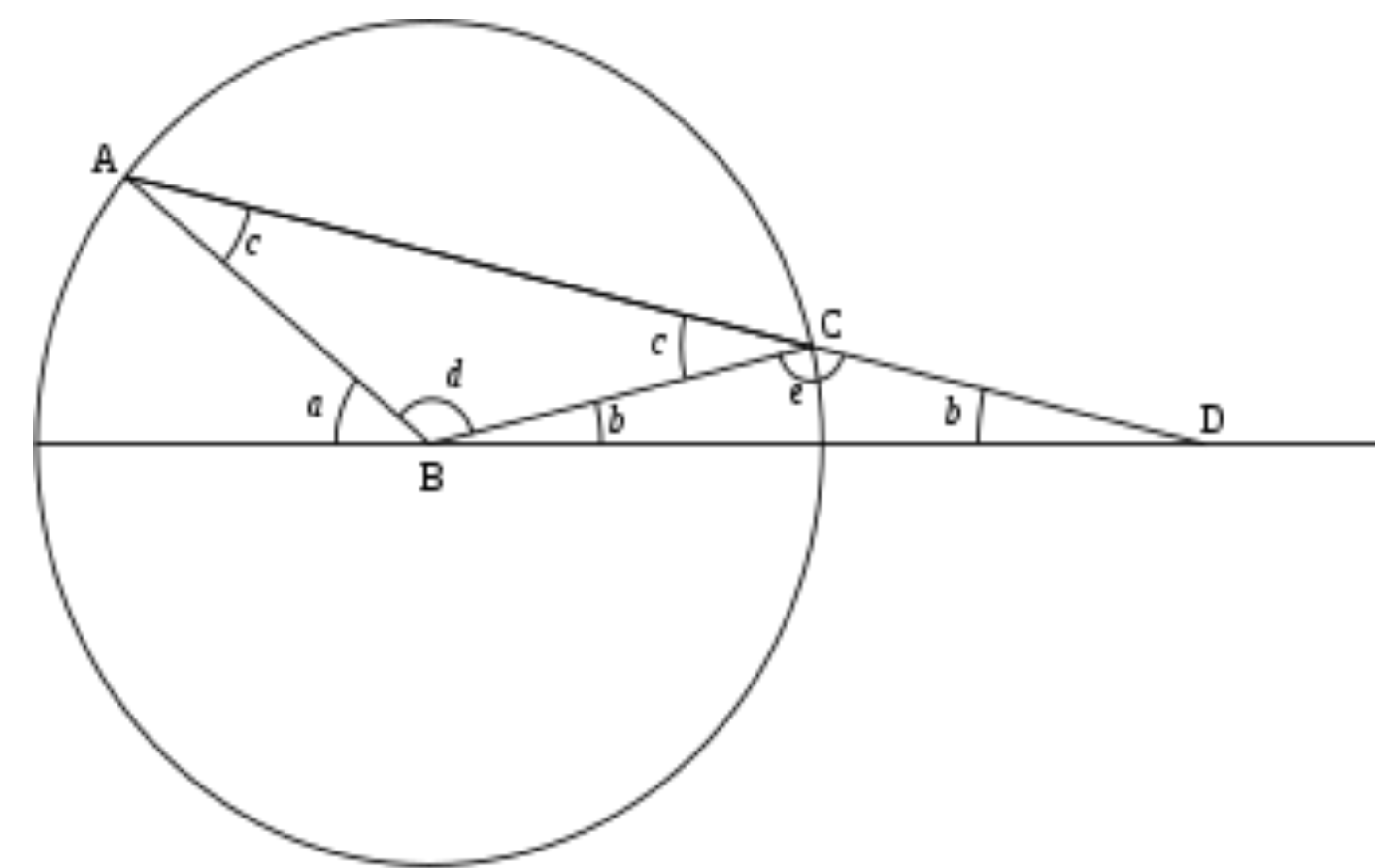
Methods

Neusis Construction

Many ancient Greek mathematicians have tried solving angle trisection problems but none of them tried solving it with ruler and compass. One of the method was neusis construction (rotating ruler) by Archimedes using marked ruler only.



Basic triangle's angle properties



Trisection of the angle using marked ruler

Hypothesis: Given AD is a straight line, and AB, BC, and CD all have equal length, angle b is equal to a/3

Proof: From 1) above $e+c=180^\circ$.

Looking at triangle BCD, from 2) $e+2b=180^\circ$.

From the last two equations, $c=2b$.

From 2) $d+2c=180^\circ$, thus $d=180^\circ-2c$, so from last, $d=180^\circ-4b$.

From 1) above, $a+d+b=180^\circ$, thus $a+(180^\circ-4b)+b=180^\circ$.

Clearing, $a-3b=0$, or $a=3b$, and the theorem is proved.

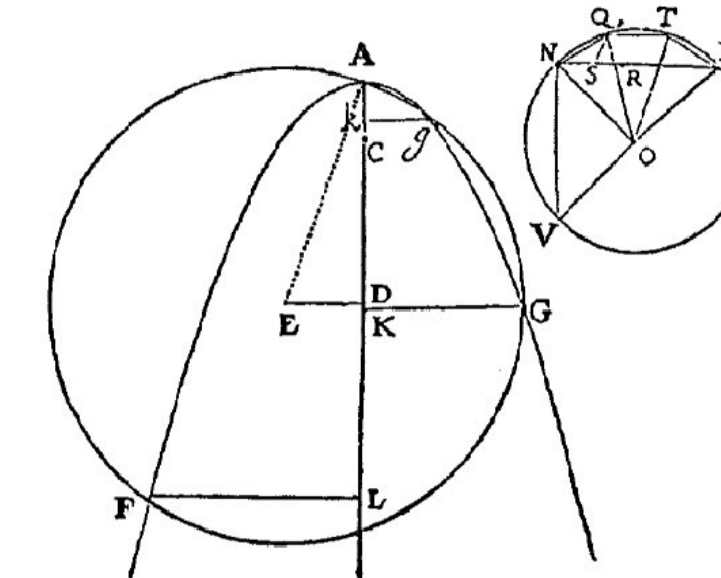
However, it is not a complete proof since Archimedes did not use the compass.

Results – Abstract Algebra

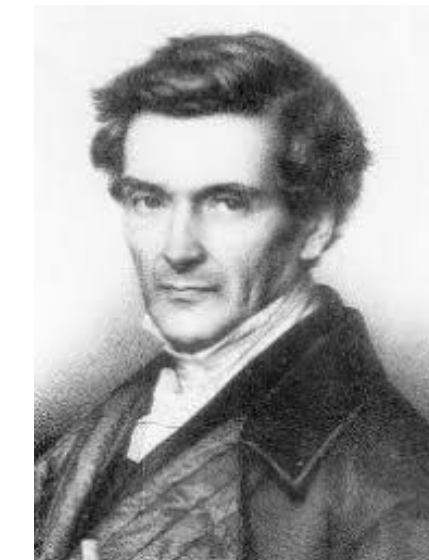


René Descartes

During 17th-18th century, many mathematicians gave an attempt to angle trisection problem. René Descartes (1596-1650) has given a proof of trisecting an angle using parabola and compass, but not a ruler.



Descartes' Solution of trisection of arbitrary angle using parabola



Pierre Wantzel

Then, French mathematician Pierre Wantzel (1814-1848) proved that it is impossible to trisect an angle in 1837 in his journal using method of algebra of a field theory. He extended ruler and compass to the algebraic fields.

First, we have to know the rational root theorem:

Theorem: If the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

has any rational roots, then they must be of the form $\pm \frac{\text{factors of } a_0}{\text{factors of } a_n}$

Then, we have two theorems

Theorem A: If a real number α is constructible using only straightedge and compass, then α satisfies a polynomial f that

- (i) has rational coefficients,
- (ii) is irreducible over the rational numbers, and
- (iii) has degree 2^k for some nonnegative integer k .

(“Constructible” means “constructible using only straightedge and compass”).

Theorem B: An angle of 20° is not constructible.

Results – cont.

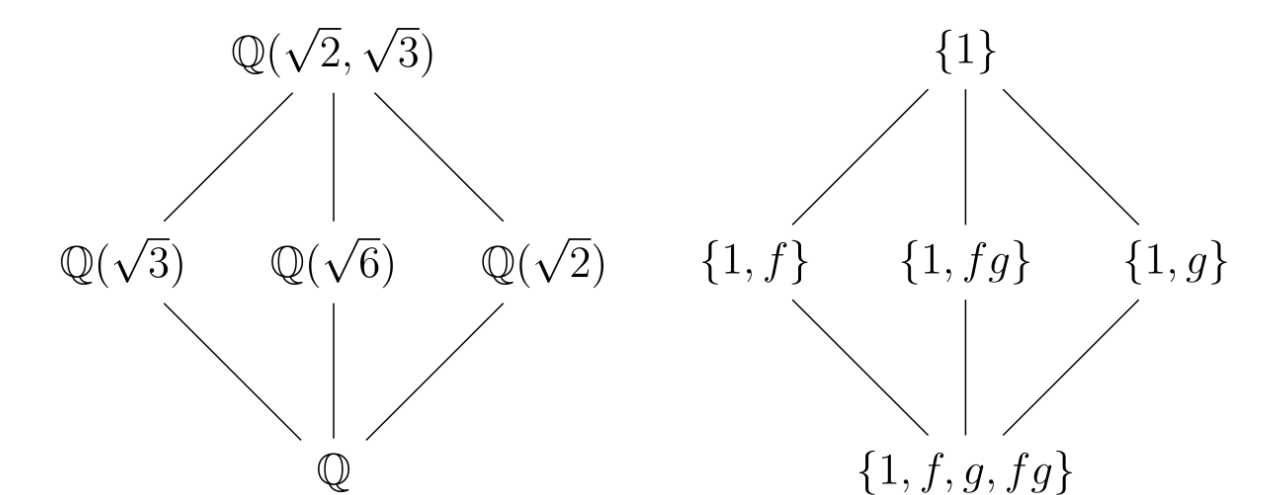
Proof : We proceed as follows:

1. An angle θ is constructible if and only if $\cos \theta$ is constructible.
2. The trigonometric identity $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$ implies that when $\alpha = \cos 20^\circ$, $4\alpha^3 - 3\alpha = \cos 60^\circ = 1/2$ or $8\alpha^3 - 6\alpha - 1 = 0$
3. The polynomial $f(x) = 8x^3 - 6x - 1$ is irreducible over \mathbb{Q} by the Rational Roots Theorem. (Specifically, we need to check that none of the possible rational roots of $f(x) = 0$, namely $x = \pm 1, \pm 1/2, \pm 1/4, \pm 1/8$, actually works.)
4. It now follows from Theorem A that $\alpha = \cos 20^\circ$, and hence 20° , is not constructible.

Conclusions

Thus, it is impossible to trisect an arbitrary angle with ruler and compass! In contrast, as shown in neusis construction method, marked straightedge (ruler that has definite length) can be used to trisect an angle without compass.

One should not forget that numerous attempts of proof were given by many great mathematicians. After finishing my research, I will do further research on Three Ancient Greek Problems. With knowledge gained, I will learn more in depth of Abstract Algebra – especially group theory, field theory and Galois theory.



Fundamental Theorem of Galois theory

References

David Poole, <https://web.maths.unsw.edu.au/~norman/papers/Trisection.pdf>
 Root theorem, <http://wmueller.com/precaculus/families/ratroot.html>
 Kim, Kwon, Lee, Park, MATH3052 Final Project(2018)

Descartes's proof - Image by Siavash CC by SA3.0
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